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COMMENT

A Mellin transform summation technique

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Abstract. A method for regularising various divergent sums is presented. The method involves the Mellin transform and is illustrated by examples.

We introduce a general method involving the Mellin transform which can be used to regularise various divergent sums which frequently appear in physics applications. The method amounts to obtaining an asymptotic expansion for $\mathrm{Tr}(\mathrm{e}^{-At})$ as $t \to 0^+$, where A is a positive definite operator with discrete eigenvalues. In a previous paper (Birmingham and Sen 1986) we have used the method to study the Candelas-Weinberg model (1984) for spaces $M^4 \times S^{2N}$ where M^4 is four-dimensional Minkowski space and S^{2N} is a $2N$-dimensional hypersphere. In this comment we will illustrate the method by way of a few simple examples.

We define the zeta function of the scalar Laplacian $A = -\nabla^2$ on S^N by

$$\zeta_A(s) \equiv \sum_{n=1}^{\infty} D_n(\Lambda_n^2)^{-s} \qquad \mathrm{Re}(s) > N/2 \tag{1}$$

where the eigenvalues and degeneracies of A are given by (Candelas and Weinberg 1984)

$$\Lambda_n^2 = n(n+N-1) \qquad D_n = \frac{(2n+N-1)(n+N-2)!}{(N-1)!\,n!}. \tag{2}$$

We note that the $n = 0$ term in (1) is omitted so that $\zeta_A(s)$ is convergent and well defined for sufficiently large s. We would like to find an analytic continuation of $\zeta_A(s)$ to negative s so that $\zeta_A(s)$ for $s = 0, -1, -2, \ldots$, makes sense. To do this we consider the following sum

$$F(t) = \mathrm{Tr}(\mathrm{e}^{-At}) \equiv \sum_{n=1}^{\infty} D_n \exp(-\Lambda_n^2 t) = \sum_{n=1}^{\infty} f(n, t). \tag{3}$$

We define the Mellin transform by

$$\tilde{f}(s) = \int_0^{\infty} x^{s-1} f(x)\,\mathrm{d}x \tag{4}$$

with the inverse transform given by

$$f(x) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-s} \tilde{f}(s)\,\mathrm{d}s. \tag{5}$$

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Performing a Mellin transform in the t-direction we have

$$\tilde{F}(s)=\sum_{n=1}^{\infty}\tilde{f}(n,s)=\zeta_A(s)\Gamma(s). \tag{6}$$

Hence

$$F(t)=\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} t^{-s}\zeta_A(s)\Gamma(s)\,\mathrm{d}s \qquad c>N/2. \tag{7}$$

Performing a Mellin transform in the n-direction we get

$$F(t)=\frac{1}{2\pi \mathrm{i}}\int_{c'-\mathrm{i}\infty}^{c'+\mathrm{i}\infty} \zeta_{\mathrm{R}}(s')\tilde{f}(s',t)\,\mathrm{d}s' \qquad c'>1 \tag{8}$$

where $\zeta_{\mathrm{R}}(s')$ is the Riemann zeta function. Now according to a theorem of Seeley (1969) the function $\zeta_A(s)$, which is only defined on the half-plane $\mathrm{Re}(s)>N/2$, actually has an analytic continuation to a meromorphic function in the whole s-plane with simple poles as singularities. These poles occur at points

$$s_j=\frac{N-j}{2} \qquad (j=0,1,2,\ldots). \tag{9}$$

Using this information together with the fact that $\Gamma(s)$ has simple poles at $s=-m$ $(m=0,1,2,\ldots)$ we can obtain from (7) an asymptotic expansion for $F(t)$ as $t\to 0^+$ of the form

$$F(t)\sim\sum_{k=-N}^{M}\alpha_{k/2}t^{k/2}+\mathrm{O}(t^{(M+1)/2}) \tag{10}$$

where the coefficients $\alpha_{k/2}$ contain geometrical information about the manifold S^N (Gilkey 1974, McKean and Singer 1967). For $k=0, 2, 4, \ldots$, we have

$$\alpha_{k/2}=[\underset{s\to -k/2}{\mathrm{residue}}\,\Gamma(s)]\lim_{s\to -k/2}\zeta_A(s). \tag{11}$$

Here we have assumed that $\zeta_A(s)\Gamma(s)$ has no double poles at $s=-k$ $(k=0,1,2,\ldots)$. This follows from Seeley's result and is verified by the absence of logarithmic terms in the expansion of $F(t)$. However, if $\tilde{f}(s',t)$ in (8) is known, we can evaluate the contour integral and obtain an alternative asymptotic expansion for $F(t)$ in terms of Riemann zeta functions. This second asymptotic expansion will then provide the required analytic continuation of $\zeta_A(s)$ to negative s. So, for example, if we are interested in finding an analytic expression for $\zeta_A(-2)$ we examine the $\mathrm{O}(t^2)$ coefficient in the asymptotic expansion. From (11) we have

$$\alpha_2=[\underset{s\to -2}{\mathrm{residue}}\,\Gamma(s)]\lim_{s\to -2}\zeta_A(s)=\tfrac{1}{2}\zeta_A(-2). \tag{12}$$

We then identify α_2 with the $\mathrm{O}(t^2)$ coefficient in the second asymptotic expansion. This will determine $\zeta_A(-2)$ in terms of Riemann zeta functions whose continuation properties are known.

As a first example we consider the case of the scalar Laplacian on S^2. We have

$$F(t)=\sum_{n=1}^{\infty}(2n+1)\,\mathrm{e}^{-n(n+1)t}=\sum_{n=1}^{\infty}f(n,t). \tag{13}$$

Now the Mellin transform of

$$\phi(n) = \mathrm{e}^{-an^2-bn} \tag{14}$$

is given by (Oberhettinger 1974)

$$\tilde{\phi}(s) = (2a)^{-s/2}\Gamma(s)\exp(b^2/8a)D_{-s}(b/(2a)^{1/2}) \tag{15}$$

where $D_a(x)$ are the parabolic cylinder functions (Abramowitz and Stegun 1970). Using (15) together with the fact that

$$M(n^\alpha\phi(n)) = \tilde{\phi}(s+\alpha) \tag{16}$$

we find

$$\begin{aligned}\tilde{f}(s,t) = {} & 2(2t)^{-(s+1)/2}\Gamma(s+1)\exp(t/8)D_{-(s+1)}((t/2)^{1/2}) \\ & +(2t)^{-s/2}\Gamma(s)\exp(t/8)D_{-s}((t/2)^{1/2}).\end{aligned} \tag{17}$$

We now insert (17) into (8) and distort the contour so as to pick up the pole contributions. In doing this we obtain an asymptotic expansion for $F(t)$ as $t \to 0^+$. If we are interested in obtaining analytic expressions for $\zeta_A(s)$ $(s=0,-1,-2)$ we need an expansion up to $O(t^2)$. This requires that we close the contour around the pole at $s=-5$. The first pole occurs at $s=1$ and corresponds to the pole of $\zeta_R(s)$. Using the properties of $D_a(x)$ (Abramowitz and Stegun 1970) we find that the contribution from the $s=1$ pole is $1/t$ while the poles at $s=(0,-1,-2,-3,-4,-5)$ contribute $O(t^0)$, $O(t^1)$, $O(t^2)$ terms. We find

$$\begin{aligned}F(t) \sim {} & 1/t - \tfrac{2}{3} - t[2\zeta_R(-3)+3\zeta_R(-2)+\zeta_R(-1)] \\ & + t^2[\zeta_R(-5)+(\tfrac{5}{2})\zeta_R(-4)+2\zeta_R(-3)+\tfrac{1}{2}\zeta_R(-2)].\end{aligned} \tag{18}$$

Using (11) we can now make the following identifications

$$\zeta_A(0) = -\tfrac{2}{3} \tag{19}$$

$$\zeta_A(-1) = [2\zeta_R(-3)+3\zeta_R(-2)+\zeta_R(-1)] = -\tfrac{1}{15} \tag{20}$$

$$\zeta_A(-2) = [2\zeta_R(-5)+5\zeta_R(-4)+4\zeta_R(-3)+\zeta_R(-2)] = \tfrac{8}{315} \tag{21}$$

where we have used the relevant properties of the Riemann zeta function (Abramowitz and Stegun 1970). We have thus achieved the required analytic continuation of $\zeta_A(s)$ to $(s=0,-1,-2)$. We note here that we can follow a similar procedure when A is the spinor Laplacian on S^N (Birmingham and Sen 1986).

Consider now the zeta function defined by Whittaker and Watson (1984)

$$\zeta(s,a) \equiv \sum_{n=0}^{\infty}(n+a)^{-s} \qquad \mathrm{Re}(s)>1. \tag{22}$$

We wish to determine $\zeta(s,a)$ for $s=0,-1,-2,\ldots$. To do this we consider the following sum

$$F(t) = \sum_{n=0}^{\infty}\mathrm{e}^{-(n+a)t} = \mathrm{e}^{-at} + \sum_{n=1}^{\infty}\mathrm{e}^{-nt}\,\mathrm{e}^{-at}. \tag{23}$$

To obtain the asymptotic expansion for $F(t)$ we first need to find an expansion for $G(t)$ defined by

$$G(t) \equiv \sum_{n=1}^{\infty}\mathrm{e}^{-nt} = \sum_{n=1}^{\infty}g(n,t). \tag{24}$$

We have (Oberhettinger 1974)

$$\tilde{g}(s, t) = t^{-s}\Gamma(s). \tag{25}$$

Proceeding as in the previous example we find

$$G(t) \sim 1/t - \tfrac{1}{2} - t\zeta_{\mathrm{R}}(-1) + \tfrac{1}{2}t^2\zeta_{\mathrm{R}}(-2) + \dots \tag{26}$$

By expanding the exponential e^{-at} in (23) and using (26) we find

$$F(t) \sim 1/t + (\tfrac{1}{2} - a) + t(\tfrac{1}{12} - a/2 + a^2/2) + t^2(-a/12 + a^2/4 - a^3/6) + \dots \tag{27}$$

We can thus identify

$$\zeta(0, a) = (\tfrac{1}{2} - a) \tag{28}$$

$$\zeta(-1, a) = (-\tfrac{1}{12} + a/2 - a^2/2) \tag{29}$$

$$\zeta(-2, a) = (-a/6 + a^2/2 - a^3/3). \tag{30}$$

The results (28)-(30) are in agreement with those of Whittaker and Watson (1984).

As a final example consider the zeta function defined by

$$\tilde{\zeta}(s, a) \equiv \sum_{n=0}^{\infty} (n^2 + a)^{-s} \qquad \mathrm{Re}(s) > \tfrac{1}{2}. \tag{31}$$

We now examine the function

$$F(t) = \sum_{n=0}^{\infty} e^{-(n^2+a)t} = e^{-at} + \sum_{n=1}^{\infty} e^{-n^2 t}\, e^{-at}. \tag{32}$$

Proceeding as above we find

$$F(t) \sim \pi^{1/2}/2t^{1/2} + \tfrac{1}{2} - \pi^{1/2}\, at^{1/2}/2 - at/2 + \pi^{1/2}a^2t^{3/2}/4 + a^2t^2/4 + \dots \tag{33}$$

We can thus identify

$$\tilde{\zeta}(0, a) = \tfrac{1}{2} \qquad \tilde{\zeta}(-1, a) = a/2 \qquad \tilde{\zeta}(-2, a) = a^2/2. \tag{34}$$

From these examples we see that the Mellin transform technique which we have introduced can provide a useful way of regularising various divergent sums. The method relies on the fact that the Mellin transform of $f(n, t)$ is known. Although this method is not the only means of regularising these sums it should be clear that it does provide an elegant and useful way of obtaining the required analytic continuation. Finally we note that the method can be used to obtain an analytic continuation of each of the zeta functions to $s = -k/2$ with $k = 0, 2, 4, \dots$. This can be seen from (11) and simply requires that we obtain the asymptotic expansion to $O(t^{k/2})$.

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